

ON THE SYMMETRY OF LOCAL CONSTANTS FOR GL_n

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ABSTRACT. Let K be a non-archimedean local field and let $G = \mathrm{GL}_n(K)$. We have shown in previous work that the smooth dual $\mathbf{Irr}(G)$ admits a complex structure: it is the disjoint union of smooth algebraic varieties, each of which is the quotient of a complex torus by a product of symmetric groups. In this article we show how the local constants interface with this complex structure. The local constants, up to a constant term, factor as characters through the corresponding complex tori. For the arithmetically unramified smooth dual of GL_n , the smooth varieties form a single extended quotient, namely $T//W$ where T is a maximal torus in the complex Langlands dual $\mathrm{GL}_n(\mathbb{C})$, and W is the Weyl group. In this case, we have explicit formulas for the local constants.

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1. INTRODUCTION

Let K be a non-archimedean local field and let $G = \mathrm{GL}_n(K)$. We have shown in previous work that the smooth dual $\mathbf{Irr}(G)$ admits a complex structure: it is the disjoint union of smooth algebraic varieties, each of which is the quotient of a complex torus by a product of symmetric groups. In this article we show how the local constants interface with this complex structure.

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We continue with some background on the local constants, which are also called *epsilon factors* [L], [D]. The epsilon factors are very important and central in the theory of Artin L -functions. If E denotes a *global* field, the completed L -function $L(s, V)$ of a representation $W_E \rightarrow \mathrm{GL}(V)$ of the Weil group W_E of the global field E defines a meromorphic function in the complex plane satisfying the functional equation

$$L(s, V) = \varepsilon(s, V) L(1 - s, V^*)$$

where V^* is the dual of the representation V of W_E , and the epsilon factor $\varepsilon(s, V)$ is defined by the product

$$(1) \quad \varepsilon(s, V) = \prod \varepsilon_{E_\nu}(s, V_\nu, \psi_\nu).$$

Here, ψ_ν is the local component at a place ν of a non-trivial additive character ψ of \mathbb{A}_K^+ trivial on E , so that ψ_ν is a non-trivial additive character of the local field E_ν . See [D, 5.11].

From now on, we will focus on the non-archimedean places of the global field E , and we will write K for the non-archimedean local field E_ν . An elementary substitution, see §2, allows on to replace the epsilon factor $\varepsilon_K(s, V, \psi)$ with three variables, by the epsilon factor $\varepsilon_K(V, \psi)$ with two variables. From now on, we will be concerned with the epsilon factor $\varepsilon_K(V, \psi)$.

From the point of view of the local Langlands correspondence for GL_n , the relevant representations are the Weil-Deligne representations, see section 2. The set $\mathcal{G}_n(K)$ of equivalence classes of n -dimensional Weil-Deligne representations can be organised as a disjoint union of complex algebraic varieties:

$$\mathcal{G}_n(K) = \bigsqcup \mathfrak{X}$$

Each variety \mathfrak{X} arises in the following way. Let ρ' denote a Weil-Deligne representation of the Weil group W_K , and let m denote the number of indecomposable summands in ρ' . Then \mathfrak{X} is the quotient of a complex torus \mathfrak{T} of dimension m by a certain finite group \mathfrak{S} :

$$\mathfrak{X} = \mathfrak{T}/\mathfrak{S}.$$

By a *rational character*, or *algebraic character*, or simply *character*, we shall mean a morphism of algebraic groups

$$\mathfrak{T} \rightarrow \mathbb{C}^\times.$$

Such a character has the form

$$(z_1, \dots, z_m) \mapsto z_1^{\beta_1} \cdots z_m^{\beta_m}$$

where the β_j are all integers.

Theorem 1.1. *Up to a constant $e(\mathfrak{X}, \psi)$, each local constant factors through a rational character $\chi(\mathfrak{X}, \psi)$ of \mathfrak{T} . Quite specifically, we have*

$$(2) \quad (z_1, \dots, z_k) \mapsto (z_1^{\beta_1}, \dots, z_k^{\beta_k})$$

where the z_j are torus coordinates, \mathfrak{X} is the orbit of the Weil-Deligne representation

$$V_1 \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus V_k \otimes \mathrm{Sp}(d_k)$$

and

$$\beta_j = (d_j - 1) \dim V_j^I + d_j[a(V_j) + n(\psi) \dim(V_j)]$$

where $a(V_j)$ denotes the Artin conductor exponent of V_j , $n(\psi)$ denotes the conductor of ψ , and I denotes the inertia subgroup of the local Weil group W_K .

Since the conductors are integers, the number β_j is an integer. So the map (2) is a rational character of \mathfrak{T} .

The rather lengthy formula for the constant $e(\mathfrak{X}, \psi)$ appears later in this article, see (19) and (20).

We emphasize that the character $\chi(\mathfrak{X}, \psi)$ and the constant $e(\mathfrak{X}, \psi)$ depend only on the irreducible component \mathfrak{X} , once the additive character ψ has been chosen and fixed.

Let T be a maximal torus in the Langlands dual group $\mathrm{GL}_n(\mathbb{C})$. We recall the idea, familiar from noncommutative geometry [Kha, p.77], of the noncommutative quotient algebra

$$\mathcal{O}(T) \rtimes W.$$

Within periodic cyclic homology (a noncommutative version of de Rham theory) there is a canonical isomorphism

$$\mathrm{HP}_*(\mathcal{O}(T) \rtimes W) \simeq H^*(T//W; \mathbb{C})$$

where

$$T//W$$

denotes the *extended quotient* of T by W , see §4. From our point of view, the extended quotient $T//W$, a complex algebraic variety, is a more concrete version of the noncommutative quotient algebra $\mathcal{O}(T) \rtimes W$.

In §4, we focus on a part of the smooth dual of $\mathrm{GL}_n(K)$, namely the arithmetically unramified smooth dual. The extended quotient $T//W$ is a model for this part of the dual, and we calculate explicitly the local constants.

We note that $\varepsilon_K(V, \psi)$ is denoted $\varepsilon_K^{\mathrm{Langlands}}(V, \psi)$ in [Ikeda] and $\varepsilon_L(V, \psi)$ in [Tate, 3.6].

In writing this article, we were greatly influenced by the preprint of Ikeda [Ikeda]. We thank Paul Baum for several valuable conversations, which led to major changes in the exposition of this Note. The main background reference is Deligne [D], but we prefer to use the notation in Langlands[L].

2. WEIL-DELIGNE REPRESENTATIONS

We need to recall some material, following closely the exposition in [BP]. Let K be a non-archimedean local field. The Weil group W_K fits into a short exact sequence

$$0 \rightarrow I_K \rightarrow W_K \xrightarrow{d} \mathbb{Z}$$

where I_K is the inertia group of K . A *Weil-Deligne representation* is a pair (ρ, N) consisting of a continuous representation $\rho : W_K \rightarrow \mathrm{GL}_n(V)$, $\dim_{\mathbb{C}}(V) = n$, together with a nilpotent endomorphism $N \in \mathrm{End}(V)$ such that

$$\rho(w)N\rho(w)^{-1} = ||w||N.$$

For any $n \geq 1$, the representation $\mathrm{Sp}(n)$ is defined by

$$V = \mathbb{C}^n = \mathbb{C}e_0 + \cdots + \mathbb{C}e_{n-1}$$

with $\rho(w)e_i = ||w||^i e_i$ and $Ne_i = e_{i+1}$ ($0 \leq i \leq n-1$), $Ne_{n-1} = 0$.

Let $\mathcal{G}_n(K)$ be the set of equivalence classes of semisimple n -dimensional Weil-Deligne representations. Let $\mathbf{Irr}(\mathrm{GL}_n(K))$ be the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(K)$.

We recall the local Langlands correspondence

$$\mathrm{rec}_K : \mathbf{Irr}(\mathrm{GL}_n(K)) \rightarrow \mathcal{G}_n(K)$$

which is unique subject to the conditions listed in [HT, p.2].

We identify the elements of the set $\mathcal{G}_1(K)$, the quasicharacters of W_K , with quasicharacters of K^\times via the local Artin reciprocity map

$$\mathrm{Art}_K : W_K \rightarrow K^\times$$

The local Langlands correspondence is compatible with twisting by quasicharacters [HT, p.2].

A quasicharacter $\psi : W_K \rightarrow \mathbb{C}^\times$ is (arithmetically) *unramified* if ψ is trivial on the inertia group I_K . In that case we have $\psi(w) = z^{d(w)}$ with $z \in \mathbb{C}^\times$. The group of unramified quasicharacters of W_K is denoted $\Psi(W_K)$. Let $\Phi = \Phi_K$ denote a geometric Frobenius element in W_K . The isomorphism $\Psi(W_K) \simeq \mathbb{C}^\times$ is secured by the map $\psi \mapsto \psi(\Phi_K)$.

Let now

$$\rho' = \rho_1 \otimes \mathrm{Sp}(r_1) \oplus \cdots \oplus \rho_m \otimes \mathrm{Sp}(r_m)$$

be a Weil-Deligne representation. The set

$$\{\psi \rho_1 \otimes \mathrm{Sp}(r_1) \oplus \cdots \oplus \psi_m \rho_m \otimes \mathrm{Sp}(r_m) : \psi_1, \dots, \psi_m \in \Psi(W_K)\}$$

will be called the *orbit* of ρ' under the action of

$$\Psi(W_K) \times \cdots \times \Psi(W_K)$$

(m factors). This orbit will be denoted $\mathcal{O}(\rho')$. The orbits create a partition of $\mathcal{G}_n(K)$. The set $\mathcal{G}_n(K)$ is a disjoint union of orbits:

$$\mathcal{G}_n(K) = \bigsqcup \mathcal{O}(\rho')$$

We note that $\Psi(W_K)^m \simeq (\mathbb{C}^\times)^m$, a complex torus. To determine the structure of each orbit, we have to pay attention to the torsion numbers of ρ_1, \dots, ρ_m and to the action of $\mathrm{GL}_n(\mathbb{C})$ by conjugation. In this way, the set $\mathcal{G}_n(K)$ acquires (locally) the structure of complex algebraic variety. Each irreducible component in this variety is the quotient of a complex torus by a product of symmetric groups.

We shall view each orbit $\mathcal{O}(\rho')$ as a pointed set, by choosing a Galois representative for each irreducible representation of W_K . We recall that, given an irreducible representation V of W_K , there exists an irreducible representation V^{Gal} of Galois type such that $V = V^{\text{Gal}} \otimes \omega_s$ for some $s \in \mathbb{C}$, see [Tate, (2.2.1)].

We will view each orbit $\mathcal{O}(\rho')$ as a pointed set, with base point

$$\rho' = \rho_1 \otimes \text{Sp}(t_1) \oplus \cdots \oplus \rho_m \otimes \text{Sp}(t_m).$$

In the notation of [Ikeda], we will choose ρ_j to be of Galois type, $\rho_j = V_j^{\text{Gal}}$.

3. THE FORMULAS

The elementary substitution referred to in the Introduction is as follows. Let

$$\varepsilon_K(s, V, \psi) = \varepsilon_K(V \otimes \omega_{s-1/2}, \psi)$$

for all $s \in \mathbb{C}$, see [Tate, (3.6.4)], [L, p.6]. For $s \in \mathbb{C}$, $\omega_s : W_K \rightarrow \mathbb{C}^\times$ is the quasicharacter defined by $\omega_s(w) = ||w||_K^s$ for all $w \in W_K$.

If V is a 1-dimensional continuous complex representation of W_K , and $\chi : W_K \rightarrow \mathbb{C}^\times$ is the corresponding quasicharacter, then $\varepsilon_K(\chi, \psi)$ is the abelian local constant of Tate, see [Tate, (3.6.3)].

We recall that, if (V, N) is any Φ -semisimple Weil-Deligne representation, then we have a finite direct sum decomposition of (V, N) into indecomposable Weil-Deligne representations as follows:

$$(3) \quad (V, N) = V_1 \otimes \text{Sp}(d_1) \oplus \cdots \oplus V_m \otimes \text{Sp}(d_m)$$

We will write

$$V_j = V_j^{\text{Gal}} \otimes \omega_{s_j}.$$

Lemma 3.1. *Let $a(V)$ denote the Artin conductor exponent of V . Then we have $a(V \otimes \omega_s) = a(V)$.*

Proof. The definition is

$$(4) \quad a(V) = \dim V - \dim V^I + \sum_{k \geq 1} \frac{1}{[I : I_k]} \cdot \dim V/V^{I_k}$$

where $I = I_0 \supset I_1 \supset \cdots \supset I_k \supset \cdots$ are the ramification subgroups of the inertia group I .

We have

$$\dim(V \otimes \omega_s) = \dim V$$

Now ω_s is an unramified quasi-character of W_K :

$$\omega_s(I) = ||\text{Art}_K(I)||^s = ||U_K||^s = 1$$

and so

$$(V \otimes \omega_s)^{I_k} = V^{I_k}$$

for all $k \geq 0$. The result now follows from (4). \square

In particular, we have

$$a(V_j) = a(V_j^{\text{Gal}}).$$

We need the following three items in order to compute epsilon factors.

3.1. Additivity. Additivity with respect to V , see [Tate, 3.4.2], [L, Theorem A (ii)]:

$$(5) \quad \varepsilon_K(V_1 \oplus \cdots \oplus V_k, \psi) = \varepsilon_K(V_1, \psi) \cdots \varepsilon_K(V_k, \psi)$$

3.2. Unramified twist. Behaviour under unramified twist, see [Tate, 3.4.5], [L, Lemma 22.4]:

$$(6) \quad \varepsilon_K(V \otimes \omega_s, \psi) = \varepsilon_K(V, \psi) q^{-s[a(V) + n(\psi) \dim V]}$$

where $a(V)$ is the Artin conductor exponent of V , and $n(\psi)$ is the conductor of ψ .

3.3. The extension formula. The extension to Weil-Deligne representations is as follows [Tate, 4.1.6]:

$$(7) \quad \varepsilon_K((V, N), \psi) := \varepsilon_K(V, \psi) \det(-\Phi|V^I/V_N^I)$$

3.4. The term $\varepsilon_K(V, \psi)$. A typical direct summand in (3) is

$$V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k$$

with $1 \leq j \leq m$, $0 \leq k \leq d_j - 1$. We have

$$V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k = V_j^{\text{Gal}} \otimes \omega_{s_j+k}$$

For this summand, we have by (6)

$$(8) \quad \varepsilon_K(V_j^{\text{Gal}} \otimes \omega_{s_j} \otimes \omega_k, \psi) = \varepsilon_K(V_j^{\text{Gal}}, \psi) q^{-(s_j+k)[a(V_j^{\text{Gal}}) + n(\psi) \dim V_j^{\text{Gal}}]}$$

and then the formula for $\varepsilon_K(V, \psi)$ follows from (5). Applying Lemma 3.1, we obtain

$$(9) \quad \varepsilon_K(V, \psi) = \prod_{j=1}^m \left(\varepsilon_K(V_j^{\text{Gal}}, \psi) \right)^{d_j} \cdot q^{-[s_j d_j + \frac{(d_j-1)d_j}{2}][a(V_j) + n(\psi) \dim(V_j)]}$$

Note that Ikeda succeeds in describing the numbers $\varepsilon_K(V_j^{\text{Gal}}, \psi)$ in terms of the non-abelian local class field theory of K , see [Ikeda, Theorem 5.4].

3.5. The determinant. The determinant is additive

$$\det(A \oplus B) = (\det A)(\det B)$$

and so it suffices to consider a typical factor in (7), namely

$$(10) \quad \det(-\Phi|E_j^I/(E_j^I)_{N_j})$$

where E_j is the W_K -module given by

$$\begin{aligned} E_j &= V_j \otimes (\omega_0 \oplus \omega_1 \oplus \cdots \oplus \omega_{d_j-1}) \\ &= (V_j^{\text{Gal}} \otimes \omega_{s_j}) \otimes (\omega_0 \oplus \omega_1 \oplus \cdots \oplus \omega_{d_j-1}) \\ &= V_j^{\text{Gal}} \otimes (\omega_{s_j} \oplus \omega_{s_j+1} \oplus \cdots \oplus \omega_{s_j+d_j-1}) \end{aligned}$$

We note that $V_j^I = (V_j^{\text{Gal}})^I$. Then the W_K -submodule fixed by the inertia group I is

$$E_j^I := V_j^I \otimes (\omega_{s_j} \oplus \omega_{s_j+1} \oplus \cdots \oplus \omega_{s_j+d_j-1})$$

The W_K -submodule of E_j annihilated by N_j is

$$(E_j)_{N_j} = V_j^{\text{Gal}} \otimes \omega_{s_j+d_j-1}$$

from which it follows that

$$(E_j)_{N_j}^I = V_j^I \otimes \omega_{s_j+d_j-1}$$

For the quotient we have the following W_K -module:

$$(11) \quad E_j^I / (E_j)_{N_j}^I \simeq V_j^I \otimes (\omega_{s_j} \oplus \cdots \oplus \omega_{s_j+d_j-2})$$

Recall that

$$\omega_s(\Phi) = \|\varpi_K\|^s = q_K^{-s}$$

It is enough to compute the action of $-\Phi$ on the W_K -module $V_j^I \otimes \omega_{s_j} \otimes \omega_k$ with $0 \leq k \leq d_j - 1$. On this W_K -module, $-\Phi$ will act as

$$q^{-(s_j+k)}(-\Phi|V_j^I)$$

and the determinant will be

$$q^{-(s_j+k) \dim V_j^I} \det(-\Phi|V_j^I)$$

There are $d_j - 1$ direct summands in (11) so the resulting determinant will be the product

$$(12) \quad \prod_{k=0}^{d_j-2} q^{-(s_j+k) \dim V_j^I} \cdot \det(-\rho_j(\Phi)|V_j^I)$$

$$(13) \quad = \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-s_j(d_j-1) \dim V_j^I} \cdot q^{-(1+2+\dots+d_j-2) \dim V_j^I}$$

$$(14) \quad = \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-s_j(d_j-1) \dim V_j^I} \cdot q^{-\frac{1}{2}(d_j-2)(d_j-1) \dim V_j^I}$$

provided that $d_j \geq 3$. By inspection, this formula is also valid for $d_j = 1$ or 2.

3.6. The term $\varepsilon_K((V, N), \psi)$. From the extension formula (7) we infer that $\varepsilon((V, N), \psi)$ is the product of (9) and (12).

We wish to isolate the term which is dependent on the variables s_1, \dots, s_m . This term is of exponential type as follows:

$$(15) \quad \text{const} \cdot q^{\sum_{j=1}^m -s_j \beta_j}$$

where

$$\beta_j = (d_j - 1) \dim V_j^I + d_j[a(V_j) + n(\psi) \dim(V_j)]$$

for all $1 \leq j \leq m$. Note that β_j is an *integer*:

$$\beta_j \in \mathbb{Z}.$$

The constant can be read off from (9) and (12). The formula (15) is intricate, but, from our point of view, it has a simple form, namely

$$(16) \quad \text{const} \cdot z_1^{\beta_1} \dots z_m^{\beta_m}$$

where

$$z_j = q^{-s_j}$$

Apart from the constant term, the formula (16) for the epsilon factor is a *rational character* of the complex torus $(\mathbb{C}^\times)^m$, i.e. the morphism

$$(\mathbb{C}^\times)^m \rightarrow \mathbb{C}^\times, \quad (z_1, \dots, z_m) \mapsto z_1^{\beta_1} \dots z_m^{\beta_m}$$

of algebraic groups.

Consider the following set:

$$(17) \quad \{\omega_{s_1} \otimes V_1^{\text{Gal}} \otimes \text{Sp}(d_1) \oplus \dots \oplus \omega_{s_m} \otimes V_m^{\text{Gal}} \otimes \text{Sp}(d_m) : s_1, \dots, s_m \in \mathbb{C}\}.$$

After allowing for conjugacy in the Langlands dual group $\text{GL}_n(\mathbb{C})$, this set has the structure of a complex algebraic variety \mathfrak{X} in \mathcal{G}_n . In fact \mathfrak{X} is an irreducible component in $\mathcal{G}_n(K)$:

$$\mathfrak{X} \subset \mathcal{G}_n(K)$$

Applying the local Langlands correspondence, we have, by transport of structure, an irreducible component in the smooth dual:

$$\text{rec}_K^{-1}(\mathfrak{X}) \subset \mathbf{Irr}(\text{GL}_n(K))$$

Looking carefully at the formulas (9) and (12), we see that the constant in (15) depends on the variety \mathfrak{X} , and on the additive character ψ . We will denote this constant by $e(\mathfrak{X}, \psi)$, so that (15) can be re-written

$$(18) \quad e(\mathfrak{X}, \psi) \cdot q^{-s_1 \beta_1} \dots q^{-s_m \beta_m}$$

which, up to the constant $e(\mathfrak{X}, \psi)$, factors as a rational character through \mathfrak{T} . The constant $e(\mathfrak{X}, \psi)$ is itself the product of

$$(19) \quad \prod_{j=1}^m \left(\varepsilon_K(V_j^{\text{Gal}}, \psi) \right)^{d_j} \cdot q^{-\frac{1}{2}(d_j-1)d_j[a(V_j)+n(\psi)\dim(V_j)]}$$

with

$$(20) \quad \prod_{j=1}^m \det(-\rho_j(\Phi)|V_j^I)^{d_j-1} \cdot q^{-\frac{1}{2}(d_j-2)(d_j-1) \dim V_j^I}$$

The terms $\varepsilon_K(V_j^{\mathrm{Gal}}, \psi)$ are the epsilon factors attached to irreducible representations of the local absolute Galois group G_K . These terms are defined in [Ikeda, p.15]. There is one case where they are readily computed.

Lemma 3.2. *Let ψ be an additive character $K \rightarrow \mathbb{C}^\times$. Then we have $\varepsilon_K(1, \psi) = 1$.*

Proof. We start with the classical formula in [Tate, 3.6.3]:

$$\varepsilon_K(\chi, \psi) = \chi(c) \frac{\int_{\mathcal{O}^\times} \chi^{-1}(u) \psi(u/c) du}{|\int_{\mathcal{O}^\times} \chi^{-1}(u) \psi(u/c) du|}$$

where c is an element of K^\times of valuation $a(\chi) + n(\psi)$. Now set $\chi = 1$ and let $n(\psi) = k$. Then we take $c = \varpi^k$. Then $u \in \mathcal{O}^\times \implies u/c \in \varpi^{-k} \mathcal{O}^\times$. But we have $\psi(\varpi^{-k} \mathcal{O}) = 1$ since ψ has conductor k . Therefore we have

$$\begin{aligned} \varepsilon_K(1, \psi) &= \frac{\int_{\mathcal{O}^\times} \psi(u/c) du}{|\int_{\mathcal{O}^\times} \psi(u/c) du|} \\ &= \frac{\mathrm{vol}(\mathcal{O}^\times)}{|\mathrm{vol}(\mathcal{O}^\times)|} \\ &= 1 \end{aligned}$$

□

4. THE ARITHMETICALLY UNRAMIFIED REPRESENTATIONS OF $\mathrm{GL}_n(K)$

Here, the underlying representation of the Weil group is the trivial n -dimensional representation $\rho : W_K \rightarrow \mathrm{GL}_n(\mathbb{C})$. So we have $V_j^{\mathrm{Gal}} = 1, 1 \leq j \leq n$.

Let W be the Weyl group \mathfrak{S}_n . The arithmetically unramified representations of $\mathrm{GL}_n(K)$ have, by definition, the following set of Langlands parameters (Weil-Deligne representations):

$$(21) \quad \{\omega_{s_1} \otimes \mathrm{Sp}(d_1) \oplus \cdots \oplus \omega_{s_k} \otimes \mathrm{Sp}(d_k) : s_j \in \mathbb{C}\}$$

where $d_1 + \cdots + d_k = n$. This set determines a complex algebraic variety \mathfrak{X} in $\mathcal{G}_n(K)$.

We choose ψ to have conductor 0, and now apply Lemma 3.2. In this case $\beta_j = d_j - 1$. We have

$$\begin{aligned} \varepsilon_K((V, N, \psi)) &= e(\mathfrak{X}, \psi) \prod_{j=1}^m q^{-(d_j-1)s_j} \\ &= e(\mathfrak{X}, \psi) \prod_{j=1}^m z_j^{d_j-1} \end{aligned}$$

where

$$e(\mathfrak{X}, \psi) = \prod_{j=1}^m (-1)^{d_j-1} q^{-(d_j-1)(d_j-2)/2}$$

and $z_j := q^{-s_j}$.

The epsilon factor records the dimensions d_j of the special representations $\mathrm{Sp}(d_j)$ which occur in the Weil-Deligne representation (V, N) .

We will now re-organise the partition $d_1 + \cdots + d_k = n$. Suppose that this partition has distinct parts t_1, \dots, t_m with $t_1 < t_2 < \cdots < t_m$ and that t_j is repeated r_j times so that

$$r_1 t_1 + \cdots + r_m t_m = n.$$

Then, as a function on the complex torus $(\mathbb{C}^\times)^{r_1 + \cdots + r_m}$, the epsilon factor is *invariant under the following product of symmetric groups*:

$$\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2} \times \cdots \times \mathfrak{S}_{r_m}$$

and therefore factors through the following quotient variety

$$(\mathbb{C}^\times)^{r_1} / \mathfrak{S}_{r_1} \times \cdots \times (\mathbb{C}^\times)^{r_m} / \mathfrak{S}_{r_m}$$

which is precisely an irreducible component of the extended quotient $T//W$. We recall the *extended quotient*. Let the finite group Γ act on the complex algebraic variety X . Let $\tilde{X} = \{(x, \gamma) : \gamma x = x\}$, let Γ act on \tilde{X} by $\gamma_1(x, \gamma) = (\gamma_1 x, \gamma_1 \gamma \gamma_1^{-1})$. Then the extended quotient of X by Γ is

$$X//\Gamma := \tilde{X}/\Gamma.$$

Let X^γ denote the γ -fixed set, and let $Z(\gamma)$ be the Γ -centralizer of γ . Choose one γ in each Γ -conjugacy class, then we have

$$X//\Gamma = \bigsqcup X^\gamma / Z(\gamma).$$

This is reminiscent of, but distinct from, the stack quotient $[X/\Gamma]$. The stack quotient $[X/\Gamma]$ is the category whose objects are the points of X and for which a morphism $x \rightarrow x'$ is given by an element $\gamma \in \Gamma$ such that $\gamma x = x'$, see [O, p.1].

Returning to the extended quotient $T//W$, every irreducible component is accounted for in this way. The epsilon factors have precisely the amount of symmetry required to factor through these quotient varieties.

EXAMPLE. Here, we consider the following Weil-Deligne representation of $\mathrm{GL}_{19}(K)$:

$$\omega_{s_1} \mathrm{Sp}(2) \oplus \omega_{s_2} \mathrm{Sp}(2) \oplus \omega_{s_3} \mathrm{Sp}(2) \oplus \omega_{s_4} \mathrm{Sp}(3) \oplus \omega_{s_5} \mathrm{Sp}(3) \oplus \omega_{s_6} \mathrm{Sp}(7)$$

The epsilon factor of this representation is

$$\mathrm{const} \cdot z_1 z_2 z_3 z_4^2 z_5^2 z_6^6$$

which will factor through the following irreducible component of the extended quotient $T//W$:

$$\mathrm{Sym}^3(\mathbb{C}^\times) \times \mathrm{Sym}^2(\mathbb{C}^\times) \times \mathbb{C}^\times$$

This perfectly illustrates the symmetry properties of the epsilon factors. Each epsilon factor has precisely the symmetry, *neither more nor less*, of the corresponding irreducible component in the extended quotient $T//W$. Each epsilon factor will therefore factor through the corresponding irreducible component in $T//W$.

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